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Technical Notes

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Buckling of a Circular Plate Weakened by Concentric Hinge or Partial Crack

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Introduction

BUCKLING of plates caused by uniform in-plane radial edge compression is an important topic in elastic stability.[1,2] There have been studies on the buckling of strengthened plates,[3] but few on weakened plates. The only related paper is from Xiang et al.,[4] who investigated a rectangular plate with an interior hinge. In the present Note we consider a circular plate partially weakened along an interior concentric circle. The results are important in several respects. First, the problem models a closed opening hatch or a circular partial crack inside a circular plate. Second, the solutions are exact, and exact solutions are useful in checking nonexact numerical methods such as the Ritz method.[5]

Formulation

Normalize all lengths by the plate radius R. The classical plate equation is

$$\nabla^2(\nabla^2 + k^2)w = 0 \tag{1}$$

where $k = R(\text{in-plane pressure/flexural rigidity})^{\frac{1}{2}}$. Letting $w = u(r)\cos n\theta$, the general solution for u can be shown to be a linear combination of r^n, r^{-n}, or $\ell n(r)$, and the Bessel functions $J_n(kr)$, $Y_n(kr)$. An aposteriori analysis shows the buckling mode is axisymmetric, and we shall henceforth consider the $n = 0$ case only.

The plate is separated into two regions. The solution in the outer region ($b \le r \le 1$), denoted by a subscript I, has the form

$$u_I(r) = AJ_0(kr) + BY_0(kr) + C + D\,\ell n(r) \tag{2}$$

where A, B, C, D are constants. The solution in the inner region ($0 \le r \le b$) bounded at the origin, denoted by the subscript II, is

$$u_{II}(r) = EJ_0(kr) + F \tag{3}$$

Along their common boundary we require the continuity of displacement

$$u_I(b) = u_{II}(b) \tag{4}$$

and bending moment

$$u_I''(b) + \nu u_I'(b)/b = u_{II}''(b) + \nu u_{II}'(b)/b \tag{5}$$

where ν is the Poisson ratio. The balance of shear gives

$$u_I'''(b) + u_I''(b)/b - u_I'(b)/b^2 + k^2u_I'(b) = u_{II}'''(b) + u_{II}''(b)/b - u_{II}'(b)/b^2 + k^2u_{II}'(b) \tag{6}$$

Here the k^2 terms are as a result of the normal component of the applied load.[6] The radial slope, however, is not continuous. We let the difference in the local angle be proportional to the moment. This is similar to a rotational spring, which models a partial crack. The relation is

$$\lambda[u_I'(b) - u_{II}'(b)] = u_{II}''(b) + \nu u_{II}'(b)/b \tag{7}$$

where $\lambda = R$ (elastic rotational spring constant)/(flexural rigidity). When $\lambda = 0$, the two regions are separated by a through crack or connected by a hinge. When $\lambda \to \infty$, the plate is undamaged. We shall consider the clamped plate where

$$u_I(1) = 0 \tag{8}$$

$$u_I'(1) = 0 \tag{9}$$

And for the simply supported plate Eq. (9) is replaced by

$$u_I''(1) + \nu u_I'(1) = 0 \tag{10}$$

For nontrivial solutions of Eqs. (4–9) or (4–8) and (10), the determinant of the coefficients yields an exact characteristic equation, which can be solved easily for the buckling parameter k by bisection.

Results and Discussion

Figure 1 shows the square root of the normalized buckling load k for the clamped plate ($\nu = 0.3$). When $\lambda = \infty$, the plate is clamped and completely continuous, and $k = 3.83171$ is the classical first root of $J_1(k) = 0$. For finite λ the buckling load goes through two dips as b is increased. The global minimum is at $b = 1$, in which case the plate is rotationally elastically restrained at the boundary, a problem first studied by Reismann[7] and Kerr.[8] The solution is the first root of

$$kJ_0(k) + (\lambda + \nu - 1)J_1(k) = 0 \tag{11}$$

When $\lambda = 0$, the buckling curve consists of two pieces. For $b < 0.535$ the curve is the same as that of the buckling of an annular plate under uniform compression, outer-edge clamped and inner-edge free.[6] In our case, the interior has no deformation (u_{II} = constant). On

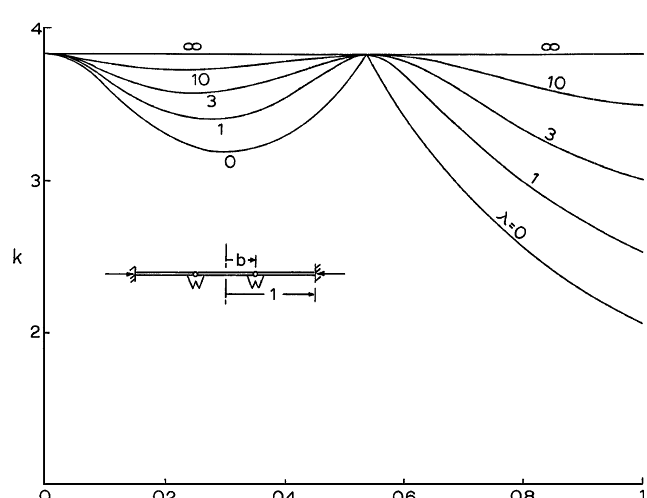

Fig. 1 Square root of the normalized bucking load k for the weakened clamped plate.

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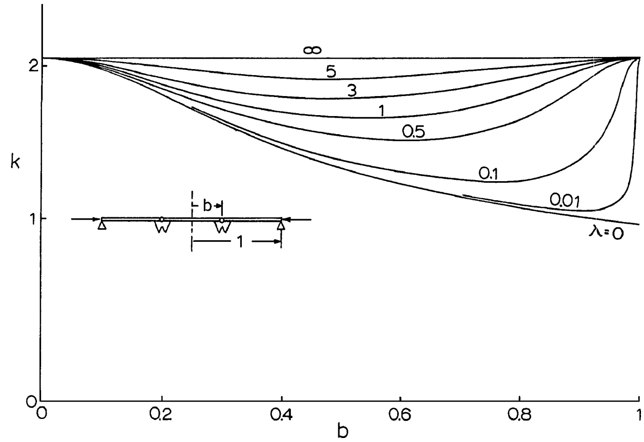

Fig. 2 Square root of the normalized buckling load *k* for the weakened simply supported plate.

the other hand, for $b > 0.535$ and $\lambda = 0$ the outer region has no deformation ($u_I =$ constant) while the inner region buckles. Using Eq. (11) with $\lambda = 0$ and $\nu = 0.3$, we find that the buckling load for a solid plate of radius one is 2.0489. Because the radius is now b, the scaled buckling load is

$$k = 2.0489/b \tag{12}$$

Of great interest is the maximum for all curves at $b = 0.535$. If a circular opening needs to be cut, then closed, then this is the radius where there is no loss of buckling load! Note this value of b is not at the inflection point ($b = 0.481$) of the continuous clamped plate.

Figure 2 shows the simply supported plate weakened on an interior circle ($\nu = 0.3$). For $\lambda = \infty$ the value of k is 2.0489 as expected. For $\lambda = 0$ the inner region is flat, and the situation is the same as the simply supported free annulus under constant compression. Yamaki[6] showed the value at $b = 1$ is $\sqrt{(1 - \nu^2)}$ or 0.9539. For intermediate λ values the buckling load is the same as the simply supported plate at $b = 0$ or 1, but is lower in between. Note the sensitivity for $b \approx 1$ and low λ, and also the singular nature as $\lambda \to 0$ as depicted by the $\lambda = 0.01$ curve.

There is much difference between a weakened plate and a strengthened plate. For example, Eq. (7) shows that the slope of the plate is discontinuous at the weakened location. The buckling load is also much lower than the continuous case, except for certain specific b values. We hope this Note will elicit more study, both theoretical and experimental, on weakened plates.

Conclusions

This work describes the effect of a circular partial crack on the buckling load of a thin circular plate. The solutions are obtained from exact characteristic determinants. It is found that the strength of the plate is greatly weakened by the partial crack. However, for a clamped plate there exists an optimum location where there is no loss of buckling load.

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